

SPECTRUMS OF EQUIVALENT SCHAUDER OPERATORS

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ABSTRACT. Assume that T_1, T_2 are equivalent Schauder operators. In this paper, we show that even in this case their Schauder spectrum may be very different in the view of operator theory. In fact, we get that if a self-adjoint Schauder operator A has more than one points in its essential spectrum $\sigma_e(A)$, then there exists a unitary spread operator U such that the Schauder spectrum $\sigma_S(UA)$ contains a ring which is depended by the essential spectrum; if there is only one point in $\sigma_e(A)$ and satisfies some conditions then there exists a unitary spread operator U such that the Schauder spectrum $\sigma_S(UA)$ contains the circumference which is depended by the essential spectrum.

1. INTRODUCTION

In their paper [3], Cao give an operator theory description of bases on a separable Hilbert space \mathcal{H} . To study operators on \mathcal{H} from a basis theory viewpoint, it is naturel to consider the behavior of operators related by equivalent bases. For examples, they show that there always be some strongly irreducible operators in the orbit of equivalent Schauder matrices([4]). However, in the usual way a spectral method consideration of operators in the equivalent orbit is also important to the joint research both on operator theory and Schauder bases. Cao introduces the conception *Schauder spectrum* to do this work. The main purpose of this paper is to show that the Schauder spectrum of Schauder operators in a given orbit can be very different.

Recall that a sequence of vectors $\{f_n\}_{n=1}^\infty$ in \mathcal{H} is said to be a *Schauder basis* [13, 9] for \mathcal{H} if every element $f \in \mathcal{H}$ has a unique series expansion $f = \sum c_n f_n$ which converges in the norm of \mathcal{H} . If $\{f_n\}$ is Schauder basic for \mathcal{H} , the *sequence space associated with $\{f_n\}$* is defined to be the linear space of all sequences $\{c_n\}$ for which $f = \sum c_n f_n$ is convergent. Two Schauder bases $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ are *equivalent* to each other if they have the same sequence space(cf, [13], definition 12.1, p131, [5], p163). Denote by ω the countable infinite cardinal. In paper [2], Cao.e.t considered the $\omega \times \omega$ matrix whose column vectors comprise a Schauder basis and call them the *Schauder matrix*. An operator has a Schauder matrix representation under some ONB is called a *Schauder operator*. Given an orthonormal basis(ONB in short) $\varphi = \{e_n\}_{n=1}^\infty$, the vector f_n in a Schauder basis sequence $\psi = \{f_n\}_{n=1}^\infty$ corresponds an l^2 sequence $\{f_{mn}\}_{m=1}^\infty$ defined uniquely by the series $f_n = \sum_{m=1}^\infty f_{mn} e_m$. The matrix $F_\psi = (f_{mn})_{\omega \times \omega}$ is called the *Schauder matrix* of basis ψ under the ONB φ .

Assume that ψ_1, ψ_2 are equivalent Schauder bases and T_{ψ_1}, T_{ψ_2} are the operators defined by Schauder matrices F_{ψ_1} and F_{ψ_2} respectively under the same ONB. Then there are no difference between ψ_1 and ψ_2 from the view of bases of the Hilbert space. Are there some notable differences between the operators T_{ψ_1} and T_{ψ_2} from the view of operator theory? From the Arsove's theorem([1], or theorem 2.12 in [2]), there is some invertible operator

$X \in L(\mathcal{H})$ such that $XT_{\psi_1} = T_{\psi_2}$ holds. Hence for a Schauder basis $\psi = \{f_n\}_{n=1}^\infty$, the set defined as

$$\mathcal{O}_{gl}(\psi) = \{X\psi; X \in gl(\mathcal{H})\}$$

in which $X\psi = \{Xf_n\}_{n=1}^\infty$ and $gl(\mathcal{H})$ consists of all invertible operators in $L(\mathcal{H})$ contains exactly all equivalent bases to ψ . Moreover, the set

$$\mathcal{O}_{gl}(F_\psi) = \{M_X F_\psi; M_X \text{ is the matrix of some operator } X \in gl(\mathcal{H})\}$$

consists of all Schauder matrix equivalent to F_ψ . In the operator level, we define

$$\mathcal{O}_{gl}(T_\psi) = \{XT_\psi; X \in gl(\mathcal{H})\}.$$

Then the set $\mathcal{O}_{gl}(T_\psi)$ consists of operators related to bases equivalent to ψ . Similarly, we consider following sets:

$$\begin{aligned} \mathcal{O}_u(\psi) &= \{U\psi; U \in U(\mathcal{H})\}, \\ \mathcal{O}_u(F_\psi) &= \{M_U F_\psi; M_U \text{ is the matrix of some unitary operator } U\}, \\ \mathcal{O}_u(T_\psi) &= \{UT_\psi; U \in U(\mathcal{H})\}, \end{aligned}$$

where $U(\mathcal{H})$ consists of all unitary operators in $L(\mathcal{H})$. Roughly speaking, by these set we bind operators related to equivalent bases of the basis ψ with the same basis const. It is easy to check that a Schauder operator T_ψ must be injective and having a dense range. Denote by $T_\psi = UA_\psi$ the polar decomposition of T_ψ , then the partial isometry U must be a unitary operator. Then the orbit $\mathcal{O}_u(T_\psi)$ is just the orbit $\mathcal{O}_u(A_\psi)$ in which A_ψ is the self-adjoint operator defined by the polar decomposition of T_ψ . In this paper we focus on unitary operators with a nice basis theory understanding, that is, a slight generalization of *spread form* defined by W. T. Gowers and B. Maurey ([6], [7]).

For a complex number λ , λ will be called in the *Schauder spectrum* of T denoted by $\sigma_S(T)$ if and only if there is no ONB such that $\lambda I - T$ has a matrix representation as a Schauder matrix. It is obviously, $\sigma(T) \supset \sigma_S(T) = \sigma_p(T) \cup \sigma_r(T)$ in which $\sigma_r(T) = \{\lambda \in \mathbb{C}, \overline{\text{Ran}(\lambda I - T)} \neq \mathcal{H}\}$.

Now we state our main theorem:

Theorem 1.1. *Assume that A is a self-adjoint Schauder operator.*

- (i) *If $\sigma(A) \subseteq [\lambda_1, \lambda_2]$, $\lambda_1 > 0$ and $\lambda_1, \lambda_2 \in \sigma_e(A)$, then there exists a unitary spread operator U such that the Schauder spectrum $\sigma_S(UA) \supseteq R$ for any rings R in the ring $R_{\lambda_1, \lambda_2}^0$;*
- (ii) *If $\lambda_1, \lambda_2 \in \sigma_e(A)$ and $0 < \lambda_1 < \lambda_2$, then there exists a unitary spread operator U such that the Schauder spectrum $\sigma_S(UA) \supseteq R$ for any rings R in the ring $R_{\lambda_1, \lambda_2}^0$;*
- (iii) *If there exists only one point $\lambda_1 \in \sigma_e(A)$, $\{t_k\}$ and $\{r_k\}$ contained in $\sigma(A)$ and satisfy that $t_k < t_{k+1}$, $r_k > r_{k+1}$, $t_k \rightarrow \lambda_1$, $r_k \rightarrow \lambda_1$, and $\sum_{n=1}^\infty \prod_{k=1}^n (\frac{t_k}{\lambda_1})^2 < \infty$, $\sum_{n=1}^\infty \prod_{k=1}^n (\frac{\lambda_1}{r_k})^2 < \infty$. Then there exists a unitary spread operator U such that the Schauder spectrum $\sigma_S(UA) \supseteq \{\lambda, |\lambda| = \lambda_1\}$.*

That is, if T is a Schauder operator, then there exist operator $T_1 \in \mathcal{O}_u(T)$ such that $\sigma_S(T_1)$ has a certain thickness. Related concept will be clear in later section.

We organize our paper as follows. In section 2, we introduce some notations and lemmas which will be used in the main theorem; in section 3, we research the case that the spectrum of self-adjoint Schauder operator has only two points; In section 4 we research the case that the essential spectrum of self-adjoint Schauder operator has only two points; In section 5, we research the case that there is no point spectrum in the spectrum of self-adjoint Schauder

operator. At last, we get that if A is a self-adjoint Schauder operator with at least two essential spectrum, then exists $UA \in \mathcal{O}_u(A)$ such that $\sigma_S(A)$ is thin and $\sigma_S(UA)$ has a certain thickness.

Remark 1.2. In the seminar held at Jilin university, Cao shows that for a Schauder operator T there must be some unitary spread U such that the Schauder operator UT has an empty Schauder spectrum. In this sense, our result in this paper show that the Schauder spectrum of UT may be very bad.

2. NOTATION AND AUXILIARY RESULTS

In this section we will introduce some notation for convenience, and some lemmas which will be used in the main theorem.

Throughout this paper, let $R_{\lambda_1, \lambda_2} = \{\lambda, \lambda_1 \leq |\lambda| \leq \lambda_2\}$, $R_{\lambda_1} = \{\lambda, |\lambda| = \lambda_1\}$, $R_{\lambda_2} = \{\lambda, |\lambda| = \lambda_2\}$ and $R_{\lambda_1, \lambda_2}^o = \{\lambda, \lambda_1 < |\lambda| < \lambda_2\}$ for $0 < \lambda_1 < \lambda_2$. If E is a subset of complex plane \mathbb{C} and $0 \notin E$, let $E^{-1} = \{\lambda, \frac{1}{\lambda} \in E\}$, $\text{Card}\{E\}$ denote the cardinal number of E .

Recall the definition of the *spread from A to B* given by W. T. Gowers and B. Maurey.

Definition 2.1. ([7], p549) Given an ONB $\{e_n\}_{n=1}^\infty$ and two infinite subsets A, B of \mathbb{N} . Let c_{00} be the vector space of all sequences of finite support. Let the elements of A and B be written in increasing order respectively as $\{a_1, a_2, \dots\}$ and $\{b_1, b_2, \dots\}$. Then e_n maps to 0 if $n \notin A$, and e_{a_k} maps to e_{b_k} for every $k \in \mathbb{N}$. Denote this map by $S_{A,B}$ and call it the spread from A to B .

Using spread forms, we can write some unitary operator into their linear combination. See the Example 4.13 in [2].

Definition 2.2. ([2], Definition 4.14) A unitary operator U on \mathcal{H} is said to be a unitary spread if there is a sequence $\{S_{A_n, B_n}\}_{n=1}^\infty$ of spreads such that the series $\sum_{n=1}^\infty S_{A_n, B_n}$ converges to U in strongly operator topology (SOT). Moreover, U will be called a finite unitary spread if U can be written as a finite linear combination.

In the paper [3], Cao.e.t proved that for each bijection σ on the set \mathbb{N} , the unitary operator U_σ is a unitary spread.

Lemma 2.3. Assume that A is a self-adjoint operator satisfying that $\sigma(A) \subseteq [\lambda_1, \lambda_2]$, $\lambda_1 > 0$, and there exists $x \neq 0$ such that $\|Ax\| = \lambda_i \|x\|$. Then $\lambda_i \in \sigma_p(A)$, and $x \in \text{Ker}(\lambda_i I - A)$, $i = 1, 2$.

Proof. Indeed we only need to prove the case of $i = 1$. The proof of the case of $i = 2$, is minor modifications of the proof of the analogous statements in the case of $i = 1$ by consider A^{-1} and will be omitted.

Since $\sigma(A) \subseteq [\lambda_1, \lambda_2]$, we know that $(Ax, x) \geq \lambda_1 \|x\|^2$ for any $x \neq 0$. Hence, $\|(\lambda_1 I - A)x\|^2 = \lambda_1^2 \|x\|^2 + \|Ax\|^2 - 2\lambda_1(Ax, x) \leq 0$, it follows that $\|(\lambda_1 I - A)x\| = 0$. That is to say $x \in \text{Ker}(\lambda_1 I - A)$ and $\lambda_1 \in \sigma_p(A)$. \square

Lemma 2.4. Assume that A is a self-adjoint operator satisfying that $\sigma(A) \subseteq [\lambda_1, \lambda_2]$, $\lambda_1 > 0$. Then, for any unitary operator U ,

- (i) $\sigma(UA) \subseteq R_{\lambda_1, \lambda_2}$;
- (ii) If $\lambda_1, \lambda_2 \notin \sigma_p(A)$, then $\sigma_p(UA) \cap R_{\lambda_i} = \emptyset$; if $\lambda_1, \lambda_2 \in \sigma_p(A)$, then $\text{Card}\{\sigma_p(UA) \cap R_{\lambda_i}\} \leq \dim \text{Ker}(\lambda_i I - A)$, $i = 1, 2$.

Proof. (i) It is well known that if T is an invertible operator, then $\sigma(T^{-1}) = \{\lambda, \lambda^{-1} \in \sigma(T)\}$, and $r(T) \leq \|T\|$ for any $T \in B(\mathcal{H})$. Thus $\sigma(UA) = \frac{1}{\sigma((UA)^{-1})}$ and $\|UA\| = \|A\| = \lambda_2$, $\|(UA)^{-1}\| = \|A^{-1}\| = \frac{1}{\lambda_1}$. It follows that $\lambda \notin \sigma(UA)$ when $|\lambda| > \lambda_2$ and $\lambda \notin \sigma((UA)^{-1})$ when $|\lambda| > \frac{1}{\lambda_1}$. Hence, $\sigma(UA) \subseteq R_{\lambda_1, \lambda_2}$, for any unitary operator U .

(ii) Indeed we only need to prove the case of $i = 1$. The proof of the case of $i = 2$, is minor modifications of the proof of the analogous statements in the case of $i = 1$ by consider A^{-1} and will be omitted.

Assume U is a unitary operator and $\lambda \in \sigma_p(UA) \cap R_{\lambda_1}$. Then there exists $x \neq 0$ such that $UAx = \lambda x$ and $\|Ax\| = \|UAx\| = \lambda_1 \|x\|$. By Lemma 2.3, $\lambda_1 \in \sigma_p(A)$ and $x \in \text{Ker}(\lambda_1 I - A)$. Hence, A and U have the matrix forms

$$A = \begin{bmatrix} \lambda_1 I & \\ & A_1 \end{bmatrix} \begin{matrix} \text{Ker}(\lambda I - UA) \\ \text{Ker}(\lambda I - UA)^\perp \end{matrix}, U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{matrix} \text{Ker}(\lambda I - UA) \\ \text{Ker}(\lambda I - UA)^\perp \end{matrix},$$

and $\text{Ker}(\lambda I - UA) \subseteq \text{Ker}(\lambda_1 I - A)$.

For any $x \in \text{Ker}(\lambda I - UA)$, $UAx = \lambda x$. Since U is a unitary operator, it is easy to check that $U_{12} = U_{21} = 0$. Hence, UA has the matrix form

$$UA = \begin{bmatrix} \lambda_1 U_{11} & \\ & U_{22} A_1 \end{bmatrix} \begin{matrix} \text{Ker}(\lambda I - UA) \\ \text{Ker}(\lambda I - UA)^\perp \end{matrix},$$

in which U_{11} and U_{22} are unitary operators and

$$\sigma_P(UA) = \sigma_P(\lambda_1 U_{11}) \cup \sigma_P(U_{22} A_1),$$

$$\text{Card}\{\sigma_P(\lambda_1 U_{11})\} \leq \dim \text{Ker}(\lambda I - UA) \leq \dim \text{Ker}(\lambda_1 I - A).$$

If there exists another $\delta \in \sigma_p(U_{22} A_1) \cap R_{\lambda_1}$, repeating the above process, we can get that $\text{Ker}(\delta I - U_{22} A_1) \subseteq \text{Ker}(\lambda_1 I - A)$ and $\text{Ker}(\delta I - U_{22} A_1) \perp \text{Ker}(\lambda I - UA)$.

Repeating the above process, we can obtain that $\text{Card}\{\sigma_p(UA) \cap R_{\lambda_1}\} \leq \dim \text{Ker}(\lambda_1 I - A)$. \square

Remark 2.5. By the above lemma, we know that if the spectrum $\sigma(A)$ of a self-adjoint Schauder operator is contained in an interval, then the Schauder spectrum $\sigma_S(UA)$ must be contained in the ring which is depended by the interval.

3. ONLY TWO POINTS IN $\sigma(A)$

In this section, we will research the case that the spectrum of self-adjoint Schauder operator A has only two points λ_1, λ_2 and $0 < \lambda_1 < \lambda_2$.

According to Lemma 2.4, we know that for any unitary operator U , there exists at most denumerable subsets σ_1 in R_{λ_1} and σ_2 in R_{λ_2} such that $\sigma_p(UA) \subseteq \sigma_1 \cup \sigma_2 \cup R_{\lambda_1, \lambda_2}^o$. In this section, we will show that if $\ker(\lambda_i - A) = \infty, i = 1, 2$, then for any at most denumerable subsets σ_1 in R_{λ_1} , σ_2 in R_{λ_2} and a ring R in $R_{\lambda_1, \lambda_2}^o$, there exists a unity operator U such that $\sigma_p(UA) \subseteq \sigma_1 \cup \sigma_2 \cup R$. Hence, there exists $UA \in \mathcal{O}_u(A)$ such that $\sigma_S(A)$ is thin and $\sigma_S(UA)$ has a certain thickness.

Lemma 3.1. *Assume that A is a self-adjoint operator satisfying that $\sigma(A) = \{\lambda_1, \lambda_2\}$, $0 < \lambda_1 < \lambda_2$ and $\dim \ker(\lambda_i - A) = \infty$. Then, there exists a unitary spread operator U such that $\sigma_p(UA) = R_{\lambda_1, \lambda_2}^o$.*

Proof. By the classical spectral theory of normal operator, we have following orthogonal decomposition of A ,

$$A = \oplus_{n \in \mathbb{Z}} A_n,$$

in which $A_0 = \lambda_2 I$, $A_{-1} = \lambda_1 I$, $A_n = \lambda_1 I$ for all $n \geq 1$ and $A_n = \lambda_2 I$ for all $n \leq -2$.

Now we choose an ONB $\{e_k^{(n)}\}_{k=1}^\infty$, for each $n \in \mathbb{Z}$. And let U be the unitary spread operator defined as

$$Ue_k^{(n)} = e_k^{(n+1)}, n \in \mathbb{Z}, k \in \mathbb{N}.$$

For a vector $x \in \mathcal{H}$ now under the ONB constructed it has a l_2 -sequence coordinate in the form

$$x = \sum_{n \in \mathbb{Z}} x^{(n)} = \sum_{n \in \mathbb{Z}} \sum_{k=1}^\infty x_k^{(n)} e_k^{(n)}.$$

Now simply we have

$$UAx = UA\left(\sum_{n \in \mathbb{Z}} x^{(n)}\right) = \sum_{n \in \mathbb{Z}} UA x^{(n)} = \sum_{n \in \mathbb{Z}} A_{n-1} x^{(n-1)}.$$

Now suppose for some $\lambda \neq 0$ we do have some vector x such that $(\lambda I - UA)x = 0$, then we have

$$\lambda x^{(n)} = A_{n-1} x^{(n-1)}.$$

Therefore, following equations hold:

$$x^{(n)} = \lambda^{-n} A_{n-1} A_{n-2} \cdots A_0 x^{(0)}, n \geq 1,$$

$$x^{(n)} = \lambda^{-n} A_n^{-1} A_{n+1}^{-1} \cdots A_{-1}^{-1} x^{(0)}, n \leq -1.$$

That is to say $x^{(n)} = \lambda^{-n} \lambda_1^{n-1} \lambda_2 x^{(0)}$, $n \geq 1$ and $x^{(n)} = \lambda^{-n} \lambda_1^{-1} \lambda_2^{n+1} x^{(0)}$, $n \leq -1$.

Since $0 < \lambda_1 < \lambda_2$ it is easy to see that if $\lambda_1 < |\lambda| < \lambda_2$ then $\sum_{n \in \mathbb{Z}} \|x^{(n)}\|^2 < \infty$; if $\lambda_1 \leq |\lambda|$ then $\|x^{(n)}\| > 1$ for $n \geq 1$, if $|\lambda| \geq \lambda_2$ then $\|x^{(-n)}\| > 1$ for $n \geq 1$, i.e. if $\lambda_1 \leq |\lambda|$ or $|\lambda| \geq \lambda_2$ then $\sum_{n \in \mathbb{Z}} \|x^{(n)}\|^2 = \infty$. Hence, $\sigma_p(UA) = \{\lambda, \lambda_1 < |\lambda| < \lambda_2\}$. □

Proposition 3.2. Assume that A is a self-adjoint operator satisfying that $\sigma(A) = \{\lambda_1, \lambda_2\}$, $0 < \lambda_1 < \lambda_2$ and $\dim \ker(\lambda_i - A) = \infty$. Then there exists a unitary spread operator U such that $\sigma_p(UA) = R$ for any rings R in the ring $R_{\lambda_1, \lambda_2}^o$.

Proof. Firstly, we prove that if $R = \{\lambda, (\lambda_1^{n_1} \lambda_2^{n_2})^{\frac{1}{n_1+n_2}} < |\lambda| < (\lambda_1^{m_1} \lambda_2^{m_2})^{\frac{1}{m_1+m_2}}\}$ is a ring in $R_{\lambda_1, \lambda_2}^o$ for some integers n_1, n_2, m_1, m_2 , then there exists a unitary spread operator U such that $\sigma_p(UA) = R$.

We assign the same notations used in the proof of Lemma 3.1.

$$x^{(n)} = \lambda^{-n} A_{n-1} A_{n-2} \cdots A_0 x^{(0)}, n \geq 1,$$

$$x^{(n)} = \lambda^{-n} A_n^{-1} A_{n+1}^{-1} \cdots A_{-1}^{-1} x^{(0)}, n \leq -1.$$

Let

$$\begin{aligned} A_0 &= A_1 = \cdots A_{n_1} = \lambda_1 I, \\ A_{n_1+1} &= A_{n_1+2} = \cdots A_{n_1+n_2} = \lambda_2 I, \\ &\vdots \end{aligned}$$

$$A_{k_1 n_1 + k_2 n_2 + k_3} = \lambda_1 I,$$

for any k_1, k_2 and $1 \leq k_3 \leq n_1$,

$$A_{k_1 n_1 + k_2 n_2 + k_3} = \lambda_2 I,$$

for any k_1, k_2 and $n_1 + 1 \leq k_3 \leq (n_1 + n_2)$.

And let

$$\begin{aligned} A_{-1} &= \cdots A_{-m_2} = \lambda_2 I, \\ A_{-m_2-1} &= A_{-m_2-2} = \cdots A_{-m_2-m_1} = \lambda_1 I, \\ &\vdots \end{aligned}$$

$$A_{-k_1 m_2 - k_2 m_1 - k_3} = \lambda_2 I,$$

for any k_1, k_2 and $1 \leq k_3 \leq m_1$,

$$A_{-k_1 m_2 - k_2 m_1 - k_3} = \lambda_1 I,$$

for any k_1, k_2 and $m_2 + 1 \leq k_3 \leq (m_1 + m_2)$.

Then we have

$$x^{(k(n_1+n_2))} = \lambda^{-k(n_1+n_2)} \lambda_1^{kn_1} \lambda_2^{kn_2} x^{(0)}, k \geq 1,$$

and

$$x^{-(k(m_1+m_2))} = \lambda^{-k(m_1+m_2)} \lambda_1^{-km_1} \lambda_2^{-km_2} x^{(0)}, k \geq 1.$$

It is easy to see that if $(\lambda_1^{n_1} \lambda_2^{n_2})^{\frac{1}{n_1+n_2}} < |\lambda| < (\lambda_1^{m_1} \lambda_2^{m_2})^{\frac{1}{m_1+m_2}}$ then $\sum_{n \in \mathbb{Z}} \|x^{(n)}\|^2 < \infty$; if $(\lambda_1^{n_1} \lambda_2^{n_2})^{\frac{1}{n_1+n_2}} \geq |\lambda|$ there exists N such that $\|x^{(n)}\| > 1$ for $n > N$, if $|\lambda| \geq (\lambda_1^{m_1} \lambda_2^{m_2})^{\frac{1}{m_1+m_2}}$ there exists N such that $\|x^{(-n)}\| > 1$ for $n > N$, i.e. if $\lambda_1 \leq |\lambda|$ or $|\lambda| \geq \lambda_2$ then $\sum_{n \in \mathbb{Z}} \|x^{(n)}\|^2 = \infty$. Hence, $\sigma_p(UA) = R$.

Now we turn to the more general situation.

Since $\lim_{n_1 \rightarrow \infty} (\lambda_1^{n_1} \lambda_2^{n_2})^{\frac{1}{n_1+n_2}} = \lambda_1$, $\lim_{n_2 \rightarrow \infty} (\lambda_1^{n_1} \lambda_2^{n_2})^{\frac{1}{n_1+n_2}} = \lambda_2$. We can get that there exists a unitary spread operator U such that $\sigma_p(UA) = R$ for any rings R in the ring $R_{\lambda_1, \lambda_2}^o$. \square

Lemma 3.3. Assume that A is a self-adjoint operator satisfying that $\sigma(A) = \{\lambda_1, \lambda_2\}$, $\lambda_1 < \lambda_2$ and $\dim \ker(\lambda_i - A) = \infty$. Then, there exists a unitary spread operator U such that $\sigma_p(UA) = \sigma_1 \cup \sigma_2$ for any at most denumerable subsets σ_1 in $\{\lambda, |\lambda| = \lambda_1\}$ and σ_2 in $\{\lambda, |\lambda| = \lambda_2\}$.

Proof. Since A is a self-adjoint operator, by the classical spectral theory of normal operator, we have following orthogonal decomposition of A

$$\begin{bmatrix} \lambda_1 I & \\ & \lambda_2 I \end{bmatrix}.$$

Let $U = U_1 \oplus U_2$, in which U_1, U_2 are unity operators such that $\sigma_p(\lambda_1 U_1) = \sigma_1, \sigma_p(\lambda_2 U_2) = \sigma_2$. Then U is a unity operator and $\sigma_p(UA) = \sigma_1 \cup \sigma_2$. \square

According to the Lemmas 2.4, 3.1, 3.3 and the Proposition 3.2, we can get the following theorem:

Theorem 3.4. *Assume that A is a self-adjoint operator satisfying that $\sigma(A) = \{\lambda_1, \lambda_2\}$, $0 < \lambda_1 < \lambda_2$ and $\dim \ker(\lambda_i - A) = \infty$. Then, there exists a unitary spread operator U such that $\sigma_p(UA) = \sigma_1 \cup \sigma_2 \cup R$ for any at most denumerable subsets σ_1 in $\{\lambda, |\lambda| = \lambda_1\}$, σ_2 in $\{\lambda, |\lambda| = \lambda_2\}$ and R is a ring in the ring $R_{\lambda_1, \lambda_2}^o$. Moreover, for any unitary operator U , there exists at most denumerable subsets σ_1 in R_{λ_1} and σ_2 in R_{λ_2} such that $\sigma_p(UA) \subseteq \sigma_1 \cup \sigma_2 \cup R_{\lambda_1, \lambda_2}^o$.*

Proof. Since A is a self-adjoint operator, by the classical spectral theory of normal operator, we have following orthogonal decomposition of $A = A_1 \oplus A_1$ where A_1 is a self-adjoint operator satisfying that $\sigma(A_1) = \{\lambda_1, \lambda_2\}$ and $\dim \ker(\lambda_i - A) = \infty$. By Lemmas 3.1, 3.3 and the Proposition 3.2, we get that there exists a unitary operator U such that $\sigma_p(UA) = \sigma_1 \cup \sigma_2 \cup R$ for any at most denumerable subsets σ_1 in $\{\lambda, |\lambda| = \lambda_1\}$, σ_2 in $\{\lambda, |\lambda| = \lambda_2\}$ and R is a ring in the ring $R_{\lambda_1, \lambda_2}^o$. The last part of this theorem is obvious by the Lemma 2.4. \square

Remark 3.5. By the above theorem, we know that if the spectrum $\sigma(A)$ of a self-adjoint Schauder operator has only two points λ_1, λ_2 and $\ker(\lambda_i - A) = \infty, i = 1, 2$, then for any ring R in $R_{\lambda_1, \lambda_2}^o$ and at most denumerable subsets σ_1 in $\{\lambda, |\lambda| = \lambda_1\}$, σ_2 in $\{\lambda, |\lambda| = \lambda_2\}$, there exists $UA \in \mathcal{O}_u(A)$ such that $\sigma_S(UA)$ contains $\sigma_1 \cup \sigma_2 \cup R$. i.e. $\sigma_S(UA)$ has a certain thickness, $\sigma_S(A)$ is thin. In other words, there is no ONB such that $\lambda I - UA$ has a matrix representation as a Schauder matrix for $\lambda \in \sigma_1 \cup \sigma_2 \cup R$.

4. ONLY TWO POINTS IN $\sigma_e(A)$

In this section, we will research the case that the essential spectrum of self-adjoint operator A has only two points λ_1, λ_2 and $0 < \lambda_1 < \lambda_2$. We will show that for any rings R in the ring $R_{\lambda_1, \lambda_2}^o = \{\lambda, |\lambda_1| < |\lambda| < \lambda_2\}$, there exists a unitary spread operator U such that $R_{\lambda_1 \lambda_2} \supseteq \sigma_p(UA) \supseteq R$. i.e. there exists $UA \in \mathcal{O}_u(A)$ such that $\sigma_S(A)$ is thin and $\sigma_S(UA)$ has a certain thickness.

Theorem 4.1. *Assume that A is a self-adjoint operator satisfying the following properties:*

(i) $\sigma(A) = \sigma_p(A) \cup \{\lambda_1, \lambda_2\}$, $0 < \lambda_1 < \lambda_2$ and λ_1, λ_2 are the unique accumulation points of $\sigma(A)$;

(ii) For each $t \in \sigma_p(A)$, $\dim \ker(A - tI) = 1$.

Then there exists a unitary spread operator U such that $R_{\lambda_1 \lambda_2} \supseteq \sigma_p(UA) \supseteq R$ for any rings R in the ring $R_{\lambda_1, \lambda_2}^o = \{\lambda, |\lambda_1| < |\lambda| < \lambda_2\}$.

Moreover, if $t_k > t_{k+1}, r_k < r_{k+1}$ for all k , then there exists a unitary spread operator U such that $\sigma_p(UA) = R$ for any rings R in the ring $R_{\lambda_1, \lambda_2}^o = \{\lambda, |\lambda_1| < |\lambda| < \lambda_2\}$; for any unitary operator U , $\sigma_p(UA) \subset R_{\lambda_1, \lambda_2} = \{\lambda, |\lambda_1| \leq |\lambda| \leq \lambda_2\}$ and $\text{Card}\{\sigma_p(UA) \cap R_{\lambda_1}\} \leq 1, \text{Card}\{\sigma_p(UA) \cap R_{\lambda_2}\} \leq 1$.

Proof. We only prove the case that $R = R_{\lambda_1 \lambda_2}$, the proof of the more general situation is similar to the Proposition 3.2 and we omit it.

The self-adjoint operator satisfying the conditions appearing in the proposition has a spectrum in the following form:

$$\sigma(A) = \{t_1, t_2, \dots, t_k, \dots\} \cup \{r_1, r_2, \dots, r_k, \dots\} \cup \{\lambda_1, \lambda_2\},$$

in which λ_1 is the accumulation point of the sequence $\{t_k\}$, λ_2 is the accumulation point of the sequence $\{r_k\}$.

Choose the subsequences $\{t_{nk}\}_{k=1}^\infty$, $n \geq 0$ of $\{t_k\}$ and $\{r_{nk}\}_{k=1}^\infty$, $n \geq 1$ of $\{r_k\}$ satisfying the following properties:

- (i) $\lim_{k \rightarrow \infty} t_{nk} = \lim_{n \rightarrow \infty} t_{nk} = \lambda_1$, $\lim_{k \rightarrow \infty} r_{nk} = \lim_{n \rightarrow \infty} r_{nk} = \lambda_2$;
- (ii) There exist t_{nk} and r_{nk} such that $t_{nk} = t_{k_0}$, $r_{nk} = r_{k_1}$ for any $t_{k_0} \in \{t_k\}$, $r_{k_1} \in \{r_k\}$;
- (iii) $t_{n_1 k_1} \neq t_{n_2 k_2}$, $r_{n_1 k_1} \neq r_{n_2 k_2}$ when $n_1 \neq n_2$ or $k_1 \neq k_2$.

Let $J_n = \{t_{nk}\}$, $n \geq 0$, $J_n = \{r_{-nk}\}$, $n \leq -1$. We rearrange these intervals as follows: $I_0 = J_0$, $I_n = J_{n+1}$ for $n \geq 1$, $I_{-1} = J_1$, $I_n = J_{n+1}$ for $n \leq -2$.

Denote $E_n = E_{I_n}$ the spectral projection on the interval I_n and by $H_n = \text{Ran}(E_n)$ for $n \in \mathbb{Z}$. Now we choose an ONB $\{e_k^{(n)}\}_{k=1}^\infty$, for each $n \in \mathbb{Z}$. Since each H_n is a reducing subspace of A , we can write A into the direct sum:

$$A = \bigoplus_{n=-\infty}^{+\infty} A_n.$$

Let $\sup_k \{t_{nk}\} = \alpha_n^{(1)}$, $\inf_k \{t_{nk}\} = \alpha_n^{(2)}$ for $n \geq 0$, $\sup_k \{r_{nk}\} = \beta_n^{(1)}$, $\inf_k \{r_{nk}\} = \beta_n^{(2)}$ for $n \geq 1$. Then $\lim_{n \rightarrow \infty} \alpha_n^{(1)} = \lim_{n \rightarrow \infty} \alpha_n^{(2)} = \lambda_1$, $\lim_{n \rightarrow \infty} \beta_n^{(1)} = \lim_{n \rightarrow \infty} \beta_n^{(2)} = \lambda_2$.

Now let U be the unitary spread operator defined as

$$Ue_k^{(n)} = e_k^{(n+1)}, n \in \mathbb{Z}, k \in \mathbb{N}.$$

For a vector $x \in \mathcal{H}$ now under the ONB constructed it has a l_2 -sequence coordinate in the form

$$x = \sum_{n \in \mathbb{Z}} x^{(n)} = \sum_{n \in \mathbb{Z}} \sum_{k=1}^{\infty} x_k^{(n)} e_k^{(n)}.$$

Now simply we have

$$UAx = UA\left(\sum_{n \in \mathbb{Z}} x^{(n)}\right) = \sum_{n \in \mathbb{Z}} UA x^{(n)} = \sum_{n \in \mathbb{Z}} A_{n-1} x^{(n-1)}.$$

Now suppose for some $\lambda \neq 0$ we do have some vector x such that $(\lambda I - UA)x = 0$, then we have

$$\lambda x^{(n)} = A_{n-1} x^{(n-1)}.$$

Therefore, following equations hold:

$$x^{(n)} = \lambda^{-n} A_{n-1} A_{n-2} \cdots A_0 x^{(0)}, n \geq 1,$$

$$x^{(n)} = \lambda^{-n} A_n^{-1} A_{n+1}^{-1} \cdots A_{-1}^{-1} x^{(0)}, n \leq -1.$$

Hence,

$$\begin{aligned} \lambda^{-n} \beta_0^{(2)} \alpha_2^{(2)} \alpha_3^{(2)} \cdots \alpha_n^{(2)} &\leq \|x^{(n)}\| \leq \lambda^{-n} \beta_0^{(1)} \alpha_2^{(1)} \alpha_3^{(1)} \cdots \alpha_n^{(1)}, n \geq 1; \\ \frac{\lambda^{-n}}{\alpha_1^{(1)} \beta_1^{(1)} \beta_2^{(1)} \cdots \beta_{-n-1}^{(1)}} &\leq \|x^{(n)}\| \leq \frac{\lambda^{-n}}{\alpha_1^{(2)} \beta_1^{(2)} \beta_2^{(2)} \cdots \beta_{-n-1}^{(2)}}, n \leq -1. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \alpha_n^{(1)} = \lim_{n \rightarrow \infty} \alpha_n^{(2)} = \lambda_1$, $\lim_{n \rightarrow \infty} \beta_n^{(1)} = \lim_{n \rightarrow \infty} \beta_n^{(2)} = \lambda_2$, it is easy to see that if $\lambda_1 < |\lambda| < \lambda_2$ then $\sum_{n \in \mathbb{Z}} \|x^{(n)}\|^2 < \infty$; if $\lambda_1 < |\lambda|$ there exists N such that $\|x^{(n)}\| > 1$

for $n > N$, if $|\lambda| > \lambda_2$ there exists N such that $\|x^{(-n)}\| > 1$ for $n > N$, i.e. if $\lambda_1 < |\lambda|$ or $|\lambda| > \lambda_2$ then $\sum_{n \in \mathbb{Z}} \|x^{(n)}\|^2 = \infty$. That is to say $\{\lambda, \lambda_1 \leq |\lambda| \leq \lambda_2\} \supseteq \sigma_p(UA) \supseteq \{\lambda, \lambda_1 < |\lambda| < \lambda_2\}$.

Moreover, if $t_k > t_{k+1}, r_k < r_{k+1}$ for all k then $t_k > \lambda_1, r_k < \lambda_2$. So $\alpha_n^{(i)} \geq \lambda_1, \beta_n^{(i)} \leq \lambda_2$ for all n and $i = 1, 2$. It is easy to see that $\sigma_p(UA) = \{\lambda, |\lambda_1| < |\lambda| < \lambda_2\}$. Furthermore, by Lemma 2.4, we get that for any unitary operator U , $\sigma_p(UA) \subset \{\lambda, |\lambda_1| \leq |\lambda| \leq \lambda_2\}$ and $\text{Card}\{\sigma_p(UA) \cap R_{\lambda_1}\} \leq 1, \text{Card}\{\sigma_p(UA) \cap R_{\lambda_2}\} \leq 1$. \square

Remark 4.2. (i) Trivial modifications adapt the proof of Theorem 4.1, we can weaken the condition $\dim \ker(A - tI) = 1$ to $\dim \ker(A - tI) < \infty$.

(ii) In the Theorem 4.1, we obtained that there exists a unitary operator U such that $\sigma_p(UA) \supseteq R$ for any rings R in the ring $R_{\lambda_1 \lambda_2}$. Moreover, we got $\sigma_p(UA) = R_{\lambda_1 \lambda_2}$ if adding the condition that $t_k > t_{k+1}, r_k < r_{k+1}$ for all k . The following examples illustrate that this condition is necessary.

Example 4.3. We assign the same notations used in the Theorem 4.1.

(1) Let $\lambda_1 = 1, \lambda_2 > 1$, and $t_{n1} = 1 - \frac{1}{n}, t_{nk} = \frac{k+n-1}{k+n} + \frac{\frac{k+n}{k+n+1} - \frac{k+n-1}{k+n}}{k+n-1} \cdot n$ for $n \geq 1, k \geq 2$ and $r_k < r_{k+1}$ for all $k \geq 1$. Then according to the proof of Theorem 4.1 and let $A_n = \bigoplus_{k=1}^{\infty} t_{nk}$, $x^{(0)} = e_0^{(0)}$ in Theorem 4.1, we obtain that $\sigma_p(UA) = \{\lambda, |\lambda_1| \leq |\lambda| < \lambda_2\}$.

(2) Let $\lambda_2 = 1, \lambda_1 < 1$, and $r_{n1} = 1 + \frac{1}{n}, r_{nk} = \frac{k+n+1}{k+n} - \frac{\frac{k+n+1}{k+n} - \frac{k+n+2}{k+n+1}}{k+n-1} \cdot n$ for $n \geq 1, k \geq 2$ and $t_k > t_{k+1}$ for all $k \geq 1$. Then according to the proof of Theorem 4.1 and let $A_n = \bigoplus_{k=1}^{\infty} t_{nk}$, $x^{(0)} = e_0^{(0)}$ in Theorem 4.1, we obtain that $\sigma_p(UA) = \{\lambda, |\lambda_1| < |\lambda| \leq \lambda_2\}$.

(3) Let $\lambda_1 = 1, \lambda_2 = 2$, and $t_{n1} = 1 - \frac{1}{n}, r_{n1} = 2 + \frac{2}{n}, t_{nk} = \frac{k+n-1}{k+n} + \frac{\frac{k+n}{k+n+1} - \frac{k+n-1}{k+n}}{k+n-1} \cdot n$, $r_{nk} = (2 + \frac{2}{k+n-1}) - \frac{(2 + \frac{2}{k+n-1}) - (2 + \frac{2}{k+n})}{k+n-1} \cdot n$ for $n \geq 1, k \geq 2$. Then according to the proof of Theorem 4.1 and let $A_n = \bigoplus_{k=1}^{\infty} t_{nk}$, $x^{(0)} = e_0^{(0)}$ in Theorem 4.1, we obtain that $\sigma_p(UA) = \{\lambda, |\lambda_1| \leq |\lambda| \leq \lambda_2\}$.

Trivial modifications adapt the proof of the Theorem 4.1, we can get the following Proposition.

Corollary 4.4. Assume that A is a self-adjoint operator satisfying the following properties:

- (i) $\sigma(A) = \sigma_p(A) \cup \{\lambda_1\}$, $0 < \lambda_1$ and λ_1 is the unique accumulation point of $\sigma(A)$;
- (ii) For each $t \in \sigma_p(A)$, $\dim \ker(A - tI) < \infty$;
- (iii) $\sigma_p(A) = \{t_1, t_2, \dots, t_k, \dots\} \cup \{r_1, r_2, \dots, r_k, \dots\}$, $t_k < t_{k+1}, r_k > r_{k+1}$, and $\sum_{n=1}^{\infty} \prod_{k=1}^n (\frac{t_k}{\lambda_1})^2 < \infty, \sum_{n=1}^{\infty} \prod_{k=1}^n (\frac{r_k}{\lambda_1})^2 < \infty$.

Then, there exists a unitary spread operator U such that $\sigma_p(UA) = \{\lambda, |\lambda| = \lambda_1\}$.

Example 4.5. Let A is a self-adjoint operator satisfying that $\sigma(A) = \sigma_p(A) \cup \{1\}$, $\sigma_p(A) = \{t_{nk}, r_{nk}\}_{k,n=1}^{\infty}$, in which $t_{n1} = 1 - \frac{1}{n}, t_{nk} = \frac{k+n-1}{k+n} + \frac{\frac{k+n}{k+n+1} - \frac{k+n-1}{k+n}}{k+n-1} \cdot n, r_{n1} = 1 + \frac{1}{n}, r_{nk} = \frac{k+n+1}{k+n} - \frac{\frac{k+n+1}{k+n} - \frac{k+n+2}{k+n+1}}{k+n-1} \cdot n$ for $n \geq 1, k \geq 2$, for each $t \in \sigma_p(A)$, $\dim \ker(A - tI) = 1$. By Corollary 4.4, and (1), (2) of Example 4.3, we can get that there exists a unitary spread operator U such that $\sigma_p(UA) = \{\lambda, |\lambda| = 1\}$.

Remark 4.6. By the Theorem 4.1, we know that if the essential spectrum of self-adjoint operator A has only two points λ_1, λ_2 and $0 < \lambda_1 < \lambda_2$ and for each $t \in \sigma_p(A)$, \dim

$\ker(A - tI) < \infty$, then for any ring R in $R_{\lambda_1, \lambda_2}^o$, there exists $UA \in \mathcal{O}_u(A)$ such that $\sigma_S(UA)$ contains R . i.e. $\sigma_S(UA)$ has a certain thickness, $\sigma_S(A)$ is thin. In other words, there is no ONB such that $\lambda I - UA$ has a matrix representation as a Schauder matrix for $\lambda \in R$.

5. NO POINTS SPECTRUM IN $\sigma(A)$

In this section, we will research the case that there is no point spectrum in $\sigma(A)$. i.e. $\sigma(A) = [\lambda_1, \lambda_2], 0 < \lambda_1$.

According to Lemma 2.4, we know that for any unitary operator U , there exists at most denumerable subsets σ_1 in R_{λ_1} and σ_2 in R_{λ_2} such that $\sigma_p(UA) \subseteq \sigma_1 \cup \sigma_2 \cup R_{\lambda_1, \lambda_2}^o$. In this section, we will show that if $\ker(\lambda_i - A) = \infty, i = 1, 2$, then for any at most denumerable subsets σ_1 in R_{λ_1} , σ_2 in R_{λ_2} and a ring R in $R_{\lambda_1, \lambda_2}^o$, there exists a unitary operator U such that $\sigma_p(UA) \subseteq \sigma_1 \cup \sigma_2 \cup R$. i.e. there exists $UA \in \mathcal{O}_u(A)$ such that $\sigma_S(A)$ is thin and $\sigma_S(UA)$ has a certain thickness.

Theorem 5.1. *Assume that A is a self-adjoint operator satisfying that $\sigma(A) = [\lambda_1, \lambda_2]$, $\lambda_1 > 0$ and $\sigma_p(A) = \emptyset$. Then, there exists a unitary spread operator U such that $\sigma_p(UA) = R$ for any rings R in the ring $R_{\lambda_1, \lambda_2}^o = \{\lambda, |\lambda_1| < |\lambda| < \lambda_2\}$.*

Proof. There is a sequence $\alpha_n \rightarrow \lambda_2$ such that $\alpha_{n+1} > \alpha_n$ for each $n \geq 1$. Moreover, the range of spectral projection $E_{[\alpha_n, \alpha_{n+1}]}$ is an infinite subspace; and a sequence $\beta_n \rightarrow \lambda_1$ such that $\beta_n > \beta_{n+1}$ for each $n \geq 1$. Moreover, the range of spectral projection $E_{[\beta_{n+1}, \beta_n]}$ is an infinite subspace.

Now we rearrange these intervals as follows.

$$\begin{aligned} J_n &= [\alpha_n, \alpha_{n+1}), n \geq 0, \\ J_n &= [\beta_{-n+1}, \beta_{-n}), n \leq -1. \end{aligned}$$

Let $I_0 = J_0, I_n = J_{-n+1}$ for $n \geq 1, I_{-1} = J_{-1}, I_n = J_{-n}$ for $n \leq -1$.

Denote $E_n = E_{I_n}$ the spectral projection on the interval I_n and by $H_n = \text{Ran}(E_n)$ for $n \in \mathbb{Z}$. Now we choose an ONB $\{e_k^{(n)}\}_{k=1}^\infty$, for each $n \in \mathbb{Z}$. Since each H_n is a reducing subspace of A , we can write A into the direct sum:

$$A = \bigoplus_{n=-\infty}^{+\infty} A_n.$$

And $\alpha_0 \|x\| \leq \|A_0 x\| \leq \alpha_1$ for $x \in H_0, \beta_0 \|x\| \leq \|A_{-1} x\| \leq \beta_{-1} \|x\|$ for $x \in H_{-1}, \beta_{-n} \|x\| \leq \|A_n x\| \leq \beta_{-n-1}$ for $x \in H_n, n \geq 1, \alpha_n \|x\| \leq \|A_0 x\| \leq \alpha_{n+1}$ for $x \in H_n, n \leq -1$.

Now let U be the unitary spread operator defined as

$$U e_k^{(n)} = e_k^{(n+1)}, n \in \mathbb{Z}, k \in \mathbb{N}.$$

For a vector $x \in \mathcal{H}$ now under the ONB constructed it has a l_2 -sequence coordinate in the form

$$x = \sum_{n \in \mathbb{Z}} x^{(n)} = \sum_{n \in \mathbb{Z}} \sum_{k=1}^{\infty} x_k^{(n)} e_k^{(n)}.$$

Now simply we have

$$U A x = U A \left(\sum_{n \in \mathbb{Z}} x^{(n)} \right) = \sum_{n \in \mathbb{Z}} U A x^{(n)} = \sum_{n \in \mathbb{Z}} A_{n-1} x^{(n-1)}.$$

Now suppose for some $\lambda \neq 0$ we do have some vector x such that $(\lambda I - UA)x = 0$, then we have

$$\lambda x^{(n)} = A_{n-1}x^{(n-1)}.$$

Therefore, following equations hold:

$$x^{(n)} = \lambda^{-n} A_{n-1} A_{n-2} \cdots A_0 x^{(0)}, n \geq 1,$$

$$x^{(n)} = \lambda^{-n} A_n^{-1} A_{n+1}^{-1} \cdots A_{-1}^{-1} x^{(0)}, n \leq -1.$$

Since $\alpha_0 \|x\| \leq \|A_0 x\| \leq \alpha_1$ for $x \in H_0$, $\beta_0 \|x\| \leq \|A_{-1} x\| \leq \beta_{-1} \|x\|$ for $x \in H_{-1}$, $\beta_{-n} \|x\| \leq \|A_n x\| \leq \beta_{-n-1}$ for $x \in H_n$, $n \geq 1$, $\alpha_n \|x\| \leq \|A_0 x\| \leq \alpha_{n+1}$ for $x \in H_n$, $n \leq -1$ and $\beta_n \rightarrow \lambda_1, \alpha_n \rightarrow \lambda_2$, it is easy to see that if $\lambda_1 < |\lambda| < \lambda_2$ then $\sum_{n \in \mathbb{Z}} \|x^{(n)}\|^2 < \infty$; if $\lambda_1 \leq |\lambda|$ there exists N such that $\|x^{(n)}\| > 1$ for $n > N$, if $|\lambda| \geq \lambda_2$ there exists N such that $\|x^{(-n)}\| > 1$ for $n > N$, i.e. if $\lambda_1 < |\lambda|$ or $|\lambda| > \lambda_2$ then $\sum_{n \in \mathbb{Z}} \|x^{(n)}\|^2 = \infty$. Hence, $\sigma_p(UA) = \{\lambda, |\lambda_1| < |\lambda| < \lambda_2\}$.

The proof of the more general situation is similar to the Proposition 3.2. \square

Remark 5.2. By the Theorem 5.1, we know that if A is a self-adjoint operator satisfying that $\sigma(A) = [\lambda_1, \lambda_2]$, $\lambda_1 > 0$ and $\sigma_p(A) = \emptyset$, then for any ring R in $R_{\lambda_1, \lambda_2}^o$, there exists $UA \in \mathcal{O}_u(A)$ such that $\sigma_S(UA)$ contains R . i.e. $\sigma_S(UA)$ has a certain thickness, $\sigma_S(A)$ is thin. In other words, there is no ONB such that $\lambda I - UA$ has a matrix representation as a Schauder matrix for $\lambda \in R$.

Trivial modifications adapt the proof of the Theorems of 3.4, 4.1 and 5.1, we can get the following proposition:

Proposition 5.3. *Assume that A is a self-adjoint operator.*

(i) *If $\sigma(A) \subseteq [\lambda_1, \lambda_2]$, $\lambda_1 > 0$ and $\lambda_1, \lambda_2 \in \sigma_e(A)$, then there exists a unitary spread operator U such that $\sigma_p(UA) = R$ for any rings R in the ring $R_{\lambda_1, \lambda_2}^o$;*

(ii) *If $\lambda_1, \lambda_2 \in \sigma_e(A)$ and $0 < \lambda_1 < \lambda_2$, then there exists a unitary spread operator U such that $\sigma_p(UA) \supseteq R$ for any rings R in the ring $R_{\lambda_1, \lambda_2}^o$. Moreover, if there exist sequence $\{t_k\}$ and $\{r_k\}$ contained in $\sigma(A)$ and satisfy that $t_k > t_{k+1}, r_k < r_{k+1}$ for all k , then there exists a unitary spread operator U such that $\sigma_p(UA) = R$ for any rings R in the ring $R_{\lambda_1, \lambda_2} = \{\lambda, |\lambda_1| < |\lambda| < \lambda_2\}$; for any unitary operator U , $\sigma_p(UA) \subset \{\lambda, |\lambda_1| \leq |\lambda| \leq \lambda_2\}$ and $\text{Card}\{\sigma_p(UA) \cap R_{\lambda_i}\} \leq \dim \text{Ker}(\lambda_i I - A), i = 1, 2$;*

(iii) *If there exists only one point $\lambda_1 \in \sigma_e(A)$, $\{t_k\}$ and $\{r_k\}$ contained in $\sigma(A)$ and satisfy that $t_k < t_{k+1}, r_k > r_{k+1}$, $t_k \rightarrow \lambda_1, r_k \rightarrow \lambda_2$, and $\sum_{n=1}^{\infty} \prod_{k=1}^n (\frac{t_k}{\lambda_1})^2 < \infty$, $\sum_{n=1}^{\infty} \prod_{k=1}^n (\frac{\lambda_1}{r_k})^2 < \infty$. Then there exists a unitary spread operator U such that $\sigma_p(UA) = \{\lambda, |\lambda| = \lambda_1\}$.*

As we know, $\sigma(T) \supset \sigma_S(T) = \sigma_p(T) \cup \{\lambda \in \mathbb{C}, \overline{\text{Ran}(\lambda I - T)} \neq \mathcal{H}\}$ for every $T \in B(\mathcal{H})$. Hence, by the Proposition 5.3, we obtain the main theorem:

Theorem 5.4. *Assume that A is a self-adjoint Schauder operator.*

(i) *If $\sigma(A) \subseteq [\lambda_1, \lambda_2]$, $\lambda_1 > 0$ and $\lambda_1, \lambda_2 \in \sigma_e(A)$, then there exists a unitary spread operator U such that the Schauder spectrum $\sigma_S(UA) \supseteq R$ for any rings R in the ring $R_{\lambda_1, \lambda_2}^o$;*

(ii) *If $\lambda_1, \lambda_2 \in \sigma_e(A)$ and $0 < \lambda_1 < \lambda_2$, then there exists a unitary spread operator U such that the Schauder spectrum $\sigma_S(UA) \supseteq R$ for any rings R in the ring $R_{\lambda_1, \lambda_2}^o$;*

(iii) If there exists only one point $\lambda_1 \in \sigma_e(A)$, $\{t_k\}$ and $\{r_k\}$ contained in $\sigma(A)$ and satisfy that $t_k < t_{k+1}, r_k > r_{k+1}, t_k \rightarrow \lambda_1, r_k \rightarrow \lambda_1$, and $\sum_{n=1}^{\infty} \prod_{k=1}^n (\frac{t_k}{\lambda_1})^2 < \infty, \sum_{n=1}^{\infty} \prod_{k=1}^n (\frac{\lambda_1}{r_k})^2 < \infty$. Then there exists a unitary spread operator U such that the Schauder spectrum $\sigma_S(UA) \supseteq \{\lambda, |\lambda| = \lambda_1\}$.

According to the Proposition 5.3 and Theorem 5.4, we know that if a self-adjoint operator A has more than one points in its essential spectrum, then there exists a unitary spread operator U such that $\sigma_p(UA)$ contains a ring which is depended by the essential spectrum, i.e. there exists $UA \in \mathcal{O}_u(A)$ such that $\sigma_S(A)$ is thin and $\sigma_S(UA)$ has a certain thickness; if there is only one point in the essential spectrum and satisfies some conditions, then there exists a unitary spread operator U such that $\sigma_p(UA)$ contains the circumference which is depended by the essential spectrum, i.e. there exists $UA \in \mathcal{O}_u(A)$ such that $\sigma_S(A)$ is at most denumerable and $\sigma_S(UA)$ is uncountable. Furthermore, by Lemma 2.4, we know that if $\sigma_e(A)$ has only one point λ_1 and $\{t_k\}$ (or $\{r_k\}$) contained in $\sigma(A)$ and satisfy that $t_k < t_{k+1}$ (or $r_k > r_{k+1}$), $t_k \rightarrow \lambda_1$ (or $r_k \rightarrow \lambda_1$), then for any unity operator U , $\sigma_p(UA) \neq R_{\lambda_1}$. However, we don't know if there exist $\{t_k\}$ and $\{r_k\}$ contained in $\sigma(A)$ and satisfy that $t_k < t_{k+1}, r_k > r_{k+1}, t_k \rightarrow \lambda_1, r_k \rightarrow \lambda_2$, does there exist a unitary operator U such that $\sigma_p(UA) = \{\lambda, |\lambda| = \lambda_1\}$. It is easy to know that if $A = \lambda I$, then the point spectrum of UA is at most denumerable for any unitary operator. We call a normal operator A is *non-trivial*, if $A \neq \lambda I$ for any $\lambda \in \mathbb{C}$. Hence, we have the following question:

Question 5.5. Assume that A is a non-trivial invertible self-adjoint operator, and there exists only one point $\lambda_1 \in \sigma_e(A)$, $\{t_k\}$ and $\{r_k\}$ contained in $\sigma(A)$ and satisfy that $t_k < t_{k+1}, r_k > r_{k+1}, t_k \rightarrow \lambda_1, r_k \rightarrow \lambda_2$. Whether there must be a unity operator U such that $\sigma_p(UA) = \{\lambda, |\lambda| = \lambda_1\}$?

REFERENCES

- [1] Arsove, Maynard G. Similar bases and isomorphisms in Frchet spaces. Math. Ann. 135, 1958, 283-293.
- [2] Y. Cao G. Tian and B. Z. Hou, Schauder Bases and Operator Theory, preprint. Aavailable at <http://arxiv.org/abs/1203.3603>.
- [3] Y. Cao B. Z. Hou and G. Tian, On unitary operators in spread form(in Chinese), accepted.
- [4] Y. Q. Ji G. Tian and Y. Cao, Strongly Irreducible Schauder Operators, preprint.
- [5] Garling, D. J. H., Symmetric bases of locally convex spaces, Studia Math. 30, 1968, 163-181.
- [6] W. T. Gowers and B. Maurey, The unconditional basic sequence problem. J. Amer. Math. Soc. 6 (1993), no. 4, 851-874.
- [7] W. T. Gowers and B. Maurey, Banach spaces with small spaces of operators, Math. Ann. 307 (1997) no. 4, 543-568.
- [8] S. Jaffard and R. M. Young, A representation theorem for Schauder bases in hilbert space, Proc. Ame. Math. soc. 126 (1998) 553-560.
- [9] C. W. McArthur, developments in schauder basis theory, Bulletin of American Mathematical Society, 78 (1972) no. 6, 877-901.
- [10] Robert E. Megginson, An introudction to Banach Space Theory, GTM183, Springe-Verlag, 1998.
- [11] A. M. Olevskii, On operators generating conditional bases in a Hilbert space, Translated from Matem- aticheskie Zametki, Vol(12), No.1, pp. 73-84, July, 1972.
- [12] Allen L. Shields, Weighted shift operators and analytic function theory, in: Topics in Operator Theory, Math. Surveys No. 13, 49-128, Amer. Math. Soc., Providence (1974).
- [13] I. Singer, Bases in Banach Space I, Springer-verlag, 1970.

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